Confidence Intervals

Confidence Intervals for Means

Example 1
Let's consider the plastic surgery data. The objective of study was to determine how many years younger patients were perceived to be after undergoing plastic surgery. For each of the 60 subjects we have a measure of the number of years he or she looked younger post surgery.

We would like to know the true mean (average number of years) for the general population of people who undergo this procedure. Let μ denote this unknown theoretical world quantity. We can estimate μ by $\overline{X} = 7.177$ which is the average number of years based on the sample of 60 patients. Recall that $E(\overline{X}) = \mu$ and $Var(\overline{X}) = \sigma^2/n$ where σ^2 is the true variance of the number of years younger a person is perceived to be when receiving this procedure. From the Central Limit Theorem we also know the sampling distribution of $\overline{X}$

$$\overline{X} \approx N(\mu, \sigma^2/n)$$
$$\frac{\overline{X} - \mu}{\sqrt{\sigma^2/n}} \approx N(0, 1),$$

where in the second line we standardized the $N(\mu, \sigma^2/n)$ random variable into a $N(0, 1)$ random variable.

Now we use the critical value from the standard Normal distribution and rearrange the terms of the inequality

$$P\left(\left|\frac{\overline{X} - \mu}{\sqrt{\sigma^2/n}}\right| \leq 1.96\right) \doteq 0.95$$
$$P\left(\overline{X} - 1.96\sqrt{\sigma^2/n} \leq \mu \leq \overline{X} + 1.96\sqrt{\sigma^2/n}\right) \doteq 0.95$$
$$95\% \text{ CI for } \mu = \left[\overline{X} - 1.96\sqrt{\sigma^2/n}, \overline{X} + 1.96\sqrt{\sigma^2/n}\right]$$

Just like for confidence intervals for proportions we have a problem since one of the terms in the formula above, namely σ^2, is unknown. How can we estimate σ^2? We can use s which is the sample standard deviation. This statistic has the nice property that $E(s^2) = \sigma^2$.

However it is not true that $\frac{\overline{X}-\mu}{\sqrt{s^2/n}} \approx N(0, 1)$. Instead, we have $\frac{\overline{X}-\mu}{\sqrt{s^2/n}} \approx t_{n-1}$ where t_{n-1} is another continuous distribution with parameter $n - 1$ called the "degrees of freedom".

Here are a couple of graphs to introduce the t distribution. The standard Normal distribution is drawn in blue (solid line) and a t distribution with 3 degrees of freedom is drawn in red (dashed line) in Figure 1. We can see that the two distributions are fairly similar. The t distribution is just a little bit wider around the tail region.

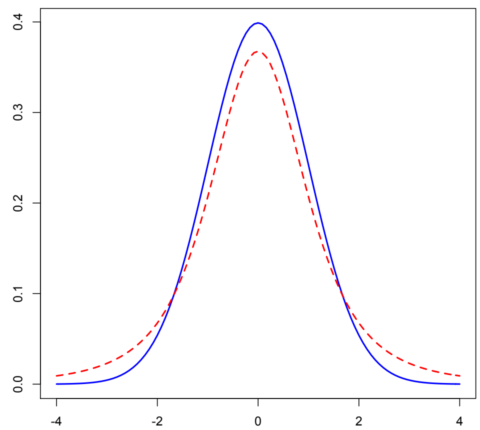

Figure 1: Comparison of the standard Normal (solid blue line) and t (dashed red line, df=3) distributions

Figure 2 shows a t distribution with 10 degrees of freedom compared to a standard Normal distribution. In this case the two distributions are even closer. When the degrees of freedom are large (corresponding to a large sample size), the t distribution becomes even more similar to the standard Normal distribution.

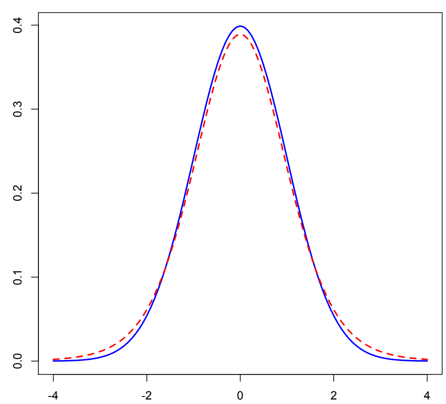

Figure 2: Comparison of the standard Normal (solid blue line) and t (dashed red line, df=10) distributions

Going back to the derivation of the confidence interval for μ, when we use s^2 instead of σ^2 we must use a critical value from a t distribution with $n-1$ degrees of freedom instead of a critical value from the standard Normal distribution. In the plastic surgery example we have 60 patients so the degrees of freedom is equal to $n-1=59$.

Table 1 summarizes some of the critical values which you can look up in a book or you can get from a statistical package for 95% confidence intervals.

Distribution	d.f. $(n-1)$	Number to use
Normal	any	1.96
	3	3.18
	6	2.45
	10	2.23
	15	2.13
	20	2.08
	30	2.04
t	40	2.02
	50	2.01
	59	2.00
	80	1.99
	100	1.98
	200	1.97
	500	1.96

Table 1: Number to use when forming 95% confidence intervals

For a t distribution the critical value for a 95% confidence interval is always slightly larger than the critical value from the standard Normal (1.96) unless the sample size is quite large. For a t distribution with 59 degrees of freedom the critical value is 2.00 which is very close to 1.96.

Returning to our calculation for the plastic surgery example ($\overline{X}=7.177$ and $s^2=8.691$)

$$\begin{aligned}
95\%\text{ CI for }\mu &= \left[\overline{X}-2.00\sqrt{s^2/n}, \overline{X}+2.00\sqrt{s^2/n}\right]\\
&= [7.177-2.00\sqrt{8.691/60}, 7.177+2.00\sqrt{8.691/60}]\\
&= [6.42, 7.94]
\end{aligned}$$

We conclude that the true mean number of years a person looks younger after a face lift procedure is between 6.42 and 7.94 years with 95% confidence.

Example 2

For a second example let's calculate a confidence interval for the mean using the skeletons data. We would like to come up with a confidence interval for the true mean of the difference between the estimated age and the actual age (μ). There are 400 observations with $\overline{X} = -14.15$ and $s^2 = 199.5$. The critical value to use in the 95% confidence interval is 1.966 and it comes from a t distribution with $n - 1 = 400 - 1 = 399$ degrees of freedom.

$$\begin{aligned} 95\% \text{ CI for } \mu &= \left[\overline{X} - 1.966\sqrt{s^2/n}, \overline{X} + 1.966\sqrt{s^2/n}\right] \\ &= [-14.15 - 1.966\sqrt{199.5/400}, -14.15 + 1.966\sqrt{199.5/400}] \\ &\doteq [-15.54, -12.76] \end{aligned}$$

With 95% confidence we can claim that the average error in age estimations for skeletons is between 12.7 and 15.5 years less than the true age.

For a 90% confidence interval the critical value from the t distribution with 399 degrees of freedom is 1.649.

$$\begin{aligned} 90\% \text{ CI for } \mu &= \left[\overline{X} - 1.649\sqrt{s^2/n}, \overline{X} + 1.649\sqrt{s^2/n}\right] \\ &= [-14.15 - 1.649\sqrt{199.5/400}, -14.15 + 1.649\sqrt{199.5/400}] \\ &\doteq [-15.31, -12.99] \end{aligned}$$