Probability: Random Variables

Normal Distributions

Some Mathematical Details

This note includes some mathematical details related to the expectation and variance of two of the results in the NORMAL DISTRIBUTIONS video.

THE EXPECTATION AND STANDARD DEVIATION OF $Z = (X - \mu)\,/\,\sigma$

If $X \sim N(\mu, \sigma)$ then $Z = (X - \mu)\,/\,\sigma \sim N(0, 1)$. To show that the expectation and standard deviation of Z are 0 and 1, respectively, we need the following facts.

For a random variable X and numerical constant a:

- $\mathrm{E}(X + a) = \mathrm{E}(X) + a$
- $\mathrm{E}(aX) = a\,\mathrm{E}(X)$
- $\mathrm{SD}(X + a) = \mathrm{SD}(X)$
- $\mathrm{SD}(aX) = a\,\mathrm{SD}(X)$, for any positive number a

Then

$$\begin{aligned}
\mathrm{E}(Z) &= \mathrm{E}\left(\frac{X - \mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\,\mathrm{E}(X - \mu) \\
&= \frac{1}{\sigma}\,(\mathrm{E}(X) - \mu) \\
&= \frac{1}{\sigma}\,(\mu - \mu) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{SD}(Z) &= \mathrm{SD}\left(\frac{X - \mu}{\sigma}\right) \\
&= \frac{1}{\sigma}\,\mathrm{SD}(X - \mu), \text{ since } \sigma > 0 \\
&= \frac{1}{\sigma}\,\mathrm{SD}(X) \\
&= \frac{1}{\sigma} \times \sigma = 1
\end{aligned}$$

The Expectation and Standard Deviation of $X_1 + X_2$ where X_1 and X_2 are independent normally distributed random variables

If $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$ are independent, then $X_1 + X_2$ has a Normal distribution. To derive the expectation and standard deviation of $X_1 + X_2$ we need the following facts.

For any two random variables X_1 and X_2:

- $\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2)$

For independent random variables X_1 and X_2:

- $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$

And for any random variable X:

- $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

Then

$$\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = \mu_1 + \mu_2$$

and

$$\mathrm{SD}(X_1 + X_2) = \sqrt{\mathrm{Var}(X_1 + X_2)} = \sqrt{\mathrm{Var}(X_1) + \mathrm{Var}(X_2)} = \sqrt{\sigma_1^2 + \sigma_2^2}$$