Linear Algebra Review[1]

Xin Bing

Department of Statistical Sciences
University of Toronto

[1] Slides adapted from Ian Goodfellow's *Deep Learning* textbook lectures

About this tutorial

- Not a comprehensive survey of all of linear algebra.
- Focused on the subset most relevant to machine learning.
- Larger subset: e.g., Linear Algebra by Gilbert Strang

Scalars

- A scalar is a single number
- Integers, real numbers, rational numbers, etc.
- Typically denoted in italic font:

$$a, n, x$$

Vectors

- A vector is an array of d numbers
- x_i be integer, real, binary, etc.
- Notation to denote type and size:

$$\boldsymbol{x} \in \mathbb{R}^d$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$$

Matrices

- A matrix is an array of numbers with two indices
- $A_{i,j}$ be integer, real, binary, etc.
- Notation to denote type and size:

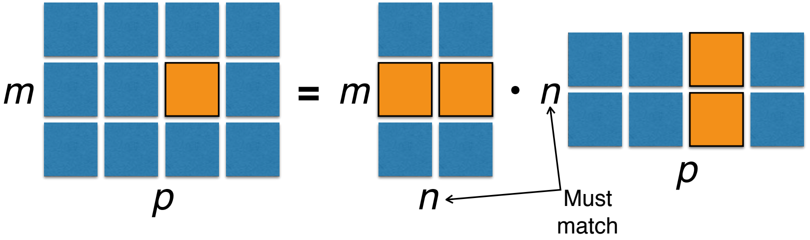
$$A \in \mathbb{R}^{m \times n}$$

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

Matrix (Dot) Product

Matrix product AB is the matrix such that

$$(AB)_{i,j} = \sum_k A_{i,k} B_{k,j}.$$



(Goodfellow 2016)

This also defines matrix-vector products $A\boldsymbol{x}$ and $\boldsymbol{x}^\top A$.

Identity Matrix

The identity matrix for $\mathbb{R}^d$ is the matrix I_d such that

$$\forall \mathbf{x} \in \mathbb{R}^d, \; I_d \mathbf{x} = \mathbf{x}$$

For example, I_3:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Transpose

The transpose of a matrix A is the matrix $A^\top$ such that $(A^\top)_{i,j} = A_{j,i}$.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \implies A^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

The transpose of a matrix can be thought of as a mirror image across the main diagonal. The transpose switches the order of the matrix product.

$$(AB)^\top = B^\top A^\top$$

Systems of equations

The matrix equation

$$A\mathbf{x} = \boldsymbol{b}$$

expands to

$$\begin{aligned}
A_{1,1}x_1 + A_{1,2}x_2 + \cdots A_{1,n}x_n &= b_1 \\
A_{2,1}x_1 + A_{2,2}x_2 + \cdots A_{2,n}x_n &= b_2 \\
&\vdots \\
A_{m,1}x_1 + A_{m,2}x_2 + \cdots A_{m,n}x_n &= b_m
\end{aligned}$$

Solving Systems of Equations

A linear system of equations can have:

- No solution
- Many solutions
- Exactly one solution, i.e. multiplying by the matrix is an invertible function

Matrix Inversion

The matrix inverse of A is the matrix A^{-1} such that

$$A^{-1}A = I_d$$

Solving a linear system using an inverse:

$$\begin{aligned} A\mathbf{x} &= \boldsymbol{b} \\ A^{-1}A\mathbf{x} &= \boldsymbol{b} \\ I_d\mathbf{x} &= A^{-1}\boldsymbol{b} \end{aligned}$$

Can be numerically unstable to implement it this way in the computer, but useful for analysis.

Invertibility

Be careful, the matrix inverse does not always exist. For example, a matrix cannot be inverted if...

- More rows than columns
- More columns than rows
- Rows or columns can be written as linear combinations of other rows or columns ("linearly dependent")

Norms

- A norm is a function that measures how "large" a vector is
- Similar to a distance between zero and the point represented by the vector

$$f(\mathbf{x}) = 0 \implies \mathbf{x} = 0$$
$$f(\mathbf{x} + \boldsymbol{y}) \leq f(\mathbf{x}) + f(\boldsymbol{y}) \text{ (the triangle inequality)}$$
$$\forall a \in \mathbb{R}, \ f(a\mathbf{x}) = |a| f(\mathbf{x})$$

Norms

- L^p norm
$$||\mathbf{x}||_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$
- Most popular norm: L2 norm, $p = 2$, i.e., the Euclidean norm.
- L1 norm:
$$||\mathbf{x}||_1 = \sum_i |x_i|$$
- Max norm, infinite norm:
$$||\mathbf{x}||_\infty = \max_i |x_i|$$

Special Matrices and Vectors

- Unit vector:

$$\|\mathbf{x}\|_2 = 1$$

- Symmetric matrix:

$$A = A^\top$$

- Orthogonal matrix

$$A^\top A = AA^\top = I_d$$
$$A^\top = A^{-1}$$

Trace

- The trace of an $n \times n$ matrix is the sum of the diagonal

$$\mathrm{Tr}(A) = \sum_i A_{i,i}$$

- It satisfies some nice commutative properties

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$$

How to learn linear algebra

- Lots of practice problems.
- Start writing out things explicitly with summations and individual indexes.
- Eventually you will be able to mostly use matrix and vector product notation quickly and easily.